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All Hermitian Hamiltonians have parity

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Abstract

It is shown that if a Hamiltonian H is Hermitian, then there always exists an operator $\mathcal{P}$ having the following properties: (i) $\mathcal{P}$ is linear and Hermitian; (ii) $\mathcal{P}$ commutes with H; (iii) $\mathcal{P}^2 = 1$; (iv) the nth eigenstate of H is also an eigenstate of $\mathcal{P}$ with eigenvalue $(-1)^n$. Given these properties, it is appropriate to refer to $\mathcal{P}$ as the parity operator and to say that H has parity symmetry, even though $\mathcal{P}$ may not refer to spatial reflection. Thus, if the Hamiltonian has the form $H = p^2 + V(x)$, where $V(x)$ is real (so that H possesses time-reversal symmetry), then it immediately follows that H has $\mathcal{PT}$ symmetry. This shows that $\mathcal{PT}$ symmetry is a generalization of Hermiticity: all Hermitian Hamiltonians of the form $H = p^2 + V(x)$ have $\mathcal{PT}$ symmetry, but not all $\mathcal{PT}$-symmetric Hamiltonians of this form are Hermitian.

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The requirement that a Hamiltonian be Hermitian guarantees that the energy eigenvalues of the Hamiltonian are real. However, in 1998 [1] it was shown that a non-Hermitian Hamiltonian can still have an entirely real spectrum provided that it possesses $\mathcal{PT}$ symmetry. For example, with properly defined boundary conditions, the Sturm–Liouville differential equation eigenvalue problem associated with the non-Hermitian Hamiltonian

$$H = p^2 + x^2(\mathrm{i}x)^\nu \qquad (\nu > 0) \tag{1}$$

exhibits a spectrum that is *real and positive*. It was argued in [1] that the reality of the spectrum of H is a consequence of the unbroken $\mathcal{PT}$ symmetry of H. A complete proof that the spectrum of H is real and positive was given by Dorey *et al* [2].

In [1] it was stated that $\mathcal{PT}$ symmetry (space-time reflection symmetry) is a weaker condition than Hermiticity in the following sense. For many different Hermitian Hamiltonians, such as $H = p^2 + x^4$, $H = p^2 + x^6$, $H = p^2 + x^8$, and so on, we can construct infinite classes of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians $H = p^2 + x^4(\mathrm{i}x)^\nu$, $H = p^2 + x^6(\mathrm{i}x)^\nu$, $H = p^2 + x^8(\mathrm{i}x)^\nu$, and so on. So long as the parameter ν is real and positive ($\nu > 0$), the $\mathcal{PT}$ symmetry of each of these Hamiltonians is not spontaneously broken and the spectrum is entirely real [3].

In this paper we show that for Hamiltonians of the form $H = p^2 + V(x)$, $\mathcal{PT}$ symmetry is a generalization of Hermiticity and that the set of Hermitian Hamiltonians (for which $V(x)$

is real) is *entirely contained* within the set of $\mathcal{PT}$-symmetric Hamiltonians. That is, we will show that if a Hamiltonian of this type is Hermitian, then it possesses both parity symmetry $\mathcal{P}$ and time-reversal symmetry $\mathcal{T}$.

As an example, consider the Hermitian Hamiltonian

$$H = p^2 + x^4 + x^3. \tag{2}$$

It is obvious that this Hamiltonian is symmetric under the operation of time reversal $\mathcal{T}$, where $\mathcal{T}$ transforms $p \to -p$ and $x \to x$. (The operator $\mathcal{T}$ also transforms $\mathrm{i} \to -\mathrm{i}$, but because the Hamiltonian is real, this fact is not relevant here.) The Hamiltonian H also possesses another discrete symmetry that can be called parity. The purpose of this paper is to show how to construct such an operator for any Hermitian Hamiltonian.

Given a Hamiltonian like that in (2) one may in principle solve the time-independent Schrödinger equation

$$H\phi_n(x) = E_n\phi_n(x) \tag{3}$$

where the eigenfunctions of H are $\phi_n(x)$ and the corresponding eigenvalues are E_n. These eigenfunctions form an orthonormal set:

$$\int \mathrm{d}x\, \phi_m^*(x)\phi_n(x) = \delta_{m,n}. \tag{4}$$

From the theory of Hermitian operators we know that the eigenfunctions form a complete basis:

$$\delta(x-y) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n^*(y). \tag{5}$$

Because the coordinate-space eigenfunctions are complete we can use them to represent the Hamiltonian as a matrix in coordinate space:

$$H(x,y) = \sum_{n=0}^{\infty} E_n\phi_n(x)\phi_n^*(y). \tag{6}$$

Let us now follow the approach of [4] to construct a new operator, which we will call $\mathcal{P}(x,y)$:

$$\mathcal{P}(x,y) \equiv \sum_{n=0}^{\infty} (-1)^n\phi_n(x)\phi_n^*(y). \tag{7}$$

Observe that $\mathcal{P}$ has the following four properties: (i) The operator $\mathcal{P}$ is linear and Hermitian. (ii) $\mathcal{P}$ commutes with the Hamiltonian. (iii) $\mathcal{P}^2 = 1$; that is, in coordinate space $\int \mathrm{d}z\, \mathcal{P}(x,z)\mathcal{P}(z,y) = \delta(x-y)$. (iv) ϕ_n is an eigenfunction of $\mathcal{P}$ with eigenvalue $(-1)^n$; that is,

$$\int \mathrm{d}z\, \mathcal{P}(x,z)\phi_n(z) = (-1)^n\phi_n(x) \tag{8}$$

by virtue of orthonormality.

Based on property (i) the operator $\mathcal{P}$ is an observable, and based on property (ii) this observable is conserved (time-independent). Moreover, because of properties (iii) and (iv) the operator $\mathcal{P}$ exhibits the characteristics of the parity operator even though the Hamiltonian may not be symmetric under space reflection. We remark that if the potential $V(x)$ of the Hamiltonian $H = p^2 + V(x)$ is invariant under the transformation $x \to -x$, then the operator $\mathcal{P}(x,y)$ in (7) is just the usual parity operator $\delta(x+y)$. Note that $\mathcal{PT}$-symmetric Hamiltonians, for which $\mathcal{P}$ is a more general symmetry than space reflection are considered in [5].

We can follow this procedure for constructing many different operators that commute with the Hamiltonian. For example, we can construct a 'triparity' operator $\mathcal{Q}(x, y)$, whose cube is unity:

$$\mathcal{Q}(x, y) = \sum_{n=0}^{\infty} \omega^n \phi_n(x)\phi_n^*(y) \tag{9}$$

where $\omega = \mathrm{e}^{\pm 2\mathrm{i}\pi/3}$, so that $\omega^3 = 1$. However, this operator is not an observable because it is not Hermitian.

We have shown that if a Hamiltonian of the form $H = p^2 + V(x)$ is Hermitian then it is also $\mathcal{PT}$ symmetric. (The converse is of course not true.) Thus, $\mathcal{PT}$ symmetry is demonstrated to be a generalization of Hermiticity.

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